Introduction to the Heisenberg $XXX_{1/2}$ Spin Chain

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ABSTRACT: This thesis investigates the Heisenberg $XXX_{1/2}$ quantum spin chain. Some basic concepts of Lie groups and Lie Algebras were studied in order to describe the $XXX_{1/2}$ quantum spin chain in terms of $su(2)$ symmetry. The two-site Heisenberg spin chain as well as the three-site Heisenberg spin chain was evaluated explicitly in order to determine the energy levels of the respective systems. The algebraic Bethe Ansatz approach was studied with the goal of diagonalizing the Hamiltonian and therefore solving for the energy spectrum of the N-site problem. Lastly some applications of quantum spin chains were investigated.

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1 Introduction

The spin $\frac{1}{2}$ Heisenberg XXX spin chain model - denoted as $XXX_{1/2}$ - consists of $N$ particles arranged in a ring with each particle possessing a spin of either up or down. Neighboring particles interact via the so-called spin-spin interaction where the energy associated to them depends on whether the spins are aligned or opposite [1]. The Heisenberg model has a rich and elegant mathematical structure which serves as a framework for all integrable models. It is not just an abstract theoretical model but in fact appears to describe the dominant physical behaviour in some metals and crystals where there is some one-dimensional isotropy [2].

1.1 The General 1-D Heisenberg $XYZ_{1/2}$ Model

The Hamiltonian of the general spin $\frac{1}{2}$ one dimensional Heisenberg Model - also known as the $XYZ_{1/2}$ Heisenberg model - described in terms of the spin $\frac{1}{2}$ operators is given by

$$H = \sum_{n=1}^{N} (J_x S_n^x S_{n+1}^x + J_y S_n^y S_{n+1}^y + J_z S_n^z S_{n+1}^z), \quad \vec{S}_{N+1} = \vec{S}_1$$  \hspace{1cm} (1.1)

where $J_x, J_y, J_z$ are real constants called the coupling constants. Sometimes one subtracts a constant from this Hamiltonian so that $E = 0$ on the ground state with all spins parallel. The physical Hilbert space that these states live in is given by

$$\mathcal{H} = \prod_{n=1}^{N} \otimes \mathbb{C}^2$$  \hspace{1cm} (1.2)

In the case that $J_x = J_y$ it is usually called the $XXZ_{1/2}$ model. Furthermore in the case that $J_x = J_y = J_z = J$ the model is called the $XXX_{1/2}$ model. The $XXX_{1/2}$ model is the simplest integrable spin chain and is the main subject of this thesis. It should be noted that many generalizations are possible, for instance one can construct spin chains with spins in higher-dimensional representations ($s = 1, \frac{3}{2}, \ldots$) and even combinations of different spins.

1.2 Brief Historical Overview

Since it was introduced in 1926 by Werner Heisenberg and P.A.M Dirac, the Heisenberg model attracted much attention from the physics community. Historically Hans Bethe’s work in 1931 on the XXX model was the starting point for subsequent discoveries in this area. Bethe formulated the famous Bethe ansatz\(^2\) to find the energy spectrum of the Hamiltonian of a one-dimensional system. In literature, his approach is nowadays referred to as

\(^{1}\)The periodicity condition on the right hand side signifies the closed chain topology, which will be explained further in section 3.

\(^{2}\)An Ansatz model refers to an assumption about the form of unknown function to facilitate the solving of a problem which is later verified by results [3].
the coordinate Bethe ansatz and has been applied to numerous other quantum integrable systems \[4\]. The algebraic Bethe ansatz (which is discussed in section 7 ) is perhaps not as powerful as the coordinate Bethe ansatz but has the advantage of being more transparent. Using an extension of Bethe’s ansatz, Rodney Baxter solved the more general XYZ model in 1971. A generalized version of three-dimensional XYZ model has been shown experimentally to accurately describe ferromagnetic and antiferromagnetic phenomena in some materials.

In order to completely understand all the intricacies of this model it is required to revisit the formulation of spin and the addition of angular momentum - to which the next section is dedicated to.

## 2 Mathematical Formulation of Spin

### 2.1 Preliminaries

The theory of angular momentum, both orbital and spin is based around a set of operators \( S^x, S^y, S^z \) that have the following commutation relations :

\[
[S^x, S^y] = i\hbar S^z \quad [S^y, S^z] = i\hbar S^x \quad [S^z, S^x] = i\hbar S^y \quad (2.1)
\]

The spin observable \( \vec{S} \) is mathematically expressed by a vector \( \vec{S} = \begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix} \) called the spin vector whose components are matrices.

\[
\vec{S} = \hbar \vec{\sigma} 
\]

where the vector \( \vec{\sigma} \) - called the Pauli vector - contains the Pauli matrices \( \sigma^x, \sigma^y, \sigma^z \) :

\[
\vec{\sigma} = \begin{pmatrix} \sigma^x \\ \sigma^y \\ \sigma^z \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} 
\]

One can construct the operator \( S^2 \), which is given by :

\[
S^2 = (\vec{S})^2 = (S^x)^2 + (S^y)^2 + (S^z)^2 
\]

with

\[
S^2 = \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} 
\]

It is important to note that \( S^2 \) commutes with each of \( S^x, S^y, S^z \). More specifically since \( S^2 \) and \( S^z \) commute they have a complete set of common eigenfunctions \(^3\) i.e. one can

\(^3\)The statement that two operators, say \( P \) and \( Q \) have a complete set of common eigenfunctions means that there exists a set of functions \( \{ f_n \} \) which are eigenfunctions of both \( P \) and \( Q \) and they are complete in the sense that any \( g \in \mathcal{H} \) can be expressed as a linear combination of \( f_n \)'s
construct simultaneous eigenstates of these operators. One can denote these eigenstates by \(|s m_s⟩\) where \(s\) is called the spin and is an integer or half-integer \((s \geq 0)\) while \(m_s\) is called the z-projection of the spin and takes on all values between \(-s\) and \(s\) (inclusive) in integer steps. So for any given \(s\), there are \(2s+1\) possible values for \(m_s\). The corresponding eigenvalue equations are given by:

\[
S^2 |s m_s⟩ = ℏ^2 s(s + 1) |s m_s⟩ \quad (2.6)
\]

\[
S^z |s m_s⟩ = ℏ m_s |s m_s⟩ \quad (2.7)
\]

The simplest example of a spin system is spin \(1/2\). This is accordingly one of the main topics of focus when investigating the Heisenberg XXX\(1/2\) spin chain. The basis of eigenstates for such a system is given by \(|\frac{1}{2} \frac{1}{2}⟩\) and \(|\frac{1}{2} -\frac{1}{2}⟩\). In matrix notation these basis states are given by

\[
|\frac{1}{2} \frac{1}{2}⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\frac{1}{2} -\frac{1}{2}⟩ = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.8)
\]

### 2.2 The Separate State Representation

Suppose one has two particles, one of spin \(s_1\) and the other of spin \(s_2\). We would like to describe the possible spin states that these particles can be in. There are two approaches that can be followed. The one approach treats the spin of each particle separately while the other looks at the combined spin of the system as a whole. The former we call the separate representation approach and is the focus of this section while the latter we call the composite representation approach and will be studied in the next section.

The particle of spin \(s_1\) can be in one of \(2s_1+1\) different spin states which corresponds to each possible value for \(m_s\). Similarly for the particle with spin \(s_2\), having \(2s_2+1\) possible spin states. Treating the particles independently implies that there are \((2s_1+1)(2s_2+1)\) different possible spin states for the two particles to be in. These states are denoted by:

\[
|s_1 m_{s_1}⟩ ⊗ |s_2 m_{s_2}⟩ \quad (2.9)
\]

where the operation \(⊗\) denotes the tensor product. This indicates that we are forming a new Hilbert space for the particles’ spins as a product of each individual particle’s original Hilbert space. See appendix A and [5],[6] for a more complete description.

It is important to note that since each particle has its own Hilbert space, there are two distinct sets of spin operators - one set for each particle. It is convenient to denote these spin operators by \(S^2_{(1)}, S^α_{(1)}\) for particle 1 \(^4\) and \(S^2_{(2)}, S^α_{(2)}\) for particle 2. In particular

\[
S^2_{(1)} = S^2 ⊗ I \quad (2.10)
\]

\[
S^2_{(2)} = I ⊗ S^2 \quad (2.11)
\]

\(^4\)The superscript \(α\) is used to represent the family of operators \(S^x, S^y\) and \(S^z\).
with $I$ being the $2 \times 2$ identity matrix i.e.

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

(2.12)

The same can be said of $S_\alpha^{(1)}$ and $S_\alpha^{(2)}$.

Other useful operators are the total spin operators for the two-particle system. The total spin vector for this two-particle system can be defined as

$$
\vec{S}_{\text{total}} = \vec{S} \otimes I + I \otimes \vec{S}
$$

(2.13)

It follows that the total spin component in the z-direction ($\vec{S}_{\text{total}}$)_z is given by:

$$
(\vec{S}_{\text{total}})_z = S_z \otimes I + I \otimes S_z
$$

(2.14)

A similar definition applies for ($\vec{S}_{\text{total}}$)_x and ($\vec{S}_{\text{total}}$)_y.

Now let us act with ($\vec{S}_{\text{total}}$)_z on a generic state $|s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle$:

$$(\vec{S}_{\text{total}})_z (|s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle) = (S_z \otimes I + I \otimes S_z) |s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle
$$

$$
= (S_z |s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle) + (I \otimes S_z) (|s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle)
$$

$$
= (S_z |s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle) + m_{s_1} \hbar (|s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle)
$$

$$
= (m_{s_1} + m_{s_2}) \hbar (|s_1 m_{s_1}\rangle \otimes |s_2 m_{s_2}\rangle)
$$

(2.15)

Where eq. (2.7) was used to go from line 3 to line 4. Hence, the eigenvalue of the total spin in the z-direction is simply the sum of the eigenvalues of the spin in the z-direction of each particle, as was expected.

In view of eq. (2.4) one can define ($\vec{S}_{\text{total}}$)_z as follows:

$$
(\vec{S}_{\text{total}})_z^2 = [(\vec{S}_{\text{total}})_z]^2 + [(\vec{S}_{\text{total}})_y]^2 + [(\vec{S}_{\text{total}})_x]^2
$$

(2.16)

Now,

$$
[(\vec{S}_{\text{total}})_z]^2 = (S_z \otimes I + I \otimes S_z)^2
$$

$$
= (S_z \otimes I)(S_z \otimes I) + (S_z \otimes I)(I \otimes S_z) + (I \otimes S_z)(S_z \otimes I) + (I \otimes S_z)(I \otimes S_z)
$$

$$
= (S_z^2 \otimes I + 2S_z^2 \otimes S_z + I \otimes (S_z^2)^2
$$

(2.17)

Similarly for [(\vec{S}_{\text{total}})_x]^2 and [(\vec{S}_{\text{total}})_y]^2:

$$
[(\vec{S}_{\text{total}})_x]^2 = (S_x^2 \otimes I + 2S_x \otimes S_x + I \otimes (S_x^2)^2
$$

(2.18)
By inspecting of the table above, it becomes clear that the total z-projection grouping

\[ [(\vec{S}_{\text{total}})^2] = (S^x)^2 \otimes I + 2S^y \otimes S^y + I \otimes (S^y)^2 \]

\hline

Hence,

\[ (\vec{S}_{\text{total}})^2 = ([S^x]^2 + (S^y)^2 + S^z)^2 \otimes I + 2(S^x \otimes S^x + S^y \otimes S^y + S^z \otimes S^z) + I \otimes [(S^x)^2 + (S^y)^2 + (S^z)^2] = (\vec{S}^2 \otimes I) + 2(S^x \otimes S^x + S^y \otimes S^y + S^z \otimes S^z) + (I \otimes \vec{S}^2) \]

\[ = (\vec{S}^2 \otimes I) + 2(\vec{S} \otimes \vec{S}) + (I \otimes \vec{S}^2) \]

One can verify that \(|s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle\) are eigenstates of \(S^2\) : \(^5\)

\[ S^2_{\{1\}}(|s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle) = (S^2 \otimes I)(|s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle) = S^2 |s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle \]

It was already shown that \(|s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle\) are eigenstates of \((\vec{S}_{\text{total}})^2\). A similar procedure can be used to show that \(|s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle\) are also eigenstates of \(S^2_{\{2\}}, S^z_{\{1\}}, S^z_{\{2\}}\)

Hence \(S^2_{\{1\}}, S^2_{\{2\}}, S^z_{\{1\}}, S^z_{\{2\}}\) and \((\vec{S}_{\text{total}})^2\) constitute a complete set of commuting observables for this system.

\subsection*{2.3 The Composite State Representation}

Now we would like to describe the system of two particles, again one of spin \(s_1\) and the other of spin \(s_2\) in a way that draws attention to the composite system as opposed to the individual particles. Suppose we calculate the total z-projection of the spin for each of these states and tabulate them where it is assumed that \(s_2 \geq s_1\).

\[
\begin{align*}
    m_{\text{total}} &= s_1 + s_2 & m_{\text{total}} &= (s_1 - 1) + s_2 & \cdots & m_{\text{total}} &= -s_1 + s_2 \\
    m_{\text{total}} &= s_1 + (s_2 - 1) & m_{\text{total}} &= (s_1 - 1) + (s_2 - 1) & \cdots & m_{\text{total}} &= -s_1 + (s_2 - 1) \\
    : & & & & & \\
    : & & & & & \\
    : & & & & & \\
    m_{\text{total}} &= s_1 + (-s_2 + 1) & m_{\text{total}} &= (s_1 - 1) + (-s_2 + 1) & \cdots & m_{\text{total}} &= -s_1 + (-s_2 + 1) \\
    m_{\text{total}} &= s_1 - s_2 & m_{\text{total}} &= (s_1 - 1) - s_2 & \cdots & m_{\text{total}} &= -s_1 - s_2
\end{align*}
\]

By inspecting of the table above, it becomes clear that the total z-projection grouping combinations are:

\[
\begin{align*}
    m_{s_1 + s_2} &\in \{s_1 + s_2, s_1 + s_2 - 1, \ldots, (s_1 - s_2) + 1, s_1 - s_2, (s_1 - s_2) - 1, \ldots, -(s_1 + s_2)\} \\
    m_{(s_1 + s_2) - 1} &\in \{(s_1 + s_2) - 1, (s_1 + s_2) - 2, \ldots, s_1 - s_2, \ldots, -(s_1 + s_2) + 1\} \\
    : & \\
    m_{s_2 - s_1} &\in \{s_2 - s_1, s_2 - s_1 - 1, \ldots, -(s_2 - s_1)\}
\end{align*}
\]

\(^5\)As they should be by the construction of the states \(|s_1 \ m_{s_1}) \otimes |s_2 \ m_{s_2}\rangle\).
Hence one can conclude that the Hilbert space of the two particles of spins \( s_1 \) and \( s_2 \) can be constructed out of all the states represented by \( s = \{ (s_1 + s_2), (s_1 + s_2 - 1), \ldots, |s_2 - s_1|\} \). The Hilbert space of the two particles is given by:

\[
\left( (s_1 + s_2) m_{(s_1+s_2)} \right) \oplus \left( (s_1 + s_2 - 1) m_{(s_1+s_2-1)} \right) \oplus \cdots \oplus \left( |s_2 - s_1| m_{|s_2-s_1|} \right)
\] (2.23)

where the operation \( \oplus \) is called the direct sum. Refer to [6] for a more complete description on the direct sum operation.

It can be shown that the the basis states for the system are eigenstates of \( S^2_{(1)}, S^2_{(2)}, (\vec{S}_{\text{total}})^z \) and \( (\vec{S}_{\text{total}})^2 \).

**Example 2.3.1.** Suppose we have a system of a spin-1 particle and a spin-2 particle. We expect there to be \( (2 \times 1 + 1)(2 \times 2 + 1) = 15 \) states. The Hilbert space for this system is given by

\[
|3 \ m_{s=3}\rangle \oplus |2 \ m_{s=2}\rangle \oplus |1 \ m_{s=1}\rangle
\] (2.24)

So the basis states are \( \{3 3, 3 2, 3 1, 3 0, 3 -1, 3 -2, 3 -3, \}
\(2 2, 2 1, 2 0, 2 -1, 2 -2, 1 1, 1 0, 1 -1\} \). This is just what was required: the 15 states are made up of a 3-spin multiplet, a 2-spin multiplet and lastly a spin-1 multiplet.

### 2.4 The Addition of Angular Momentum

In the separate state representation the basis states for the system are eigenstates of \( S^2_{(1)}, S^2_{(2)}, S^z_{(1)}, S^z_{(2)} \) and \( (\vec{S}_{\text{total}})^z \) and live in a Hilbert space that is given by

\[
|s_1 \ m_{s_1}\rangle \otimes |s_2 \ m_{s_2}\rangle
\] (2.25)

In the composite state representation, the basis states for the system are eigenstates of \( S^2_{(1)}, S^2_{(2)}, (\vec{S}_{\text{total}})^z \) and \( (\vec{S}_{\text{total}})^2 \) and live in a Hilbert space given by

\[
\left( (s_1 + s_2) m_{(s_1+s_2)} \right) \oplus \left( (s_1 + s_2 - 1) m_{(s_1+s_2-1)} \right) \oplus \cdots \oplus \left( |s_2 - s_1| m_{|s_2-s_1|} \right)
\] (2.26)

These two Hilbert spaces describe the same physical system, consequently they must in fact be the same Hilbert space, i.e.

\[
|s_1 \ m_{s_1}\rangle \otimes |s_2 \ m_{s_2}\rangle = \left( (s_1 + s_2) m_{(s_1+s_2)} \right) \oplus \left( (s_1 + s_2 - 1) m_{(s_1+s2-1)} \right) \oplus \cdots \oplus \left( |s_2 - s_1| m_{|s_2-s_1|} \right)
\] (2.27)

Hence, the system can be represented in terms of the eigenstates of the the individual particle’s spin operators or in terms of the eigenstates of the system’s total spin operators. This is the central idea behind the addition of angular momentum.

#### 2.4.1 Matching the Basis States

The task here is to determine which basis states in the Separate representation are equal to which basis states in the Composite representation.
Example 2.4.1. Let us now look at a system of two spin \( \frac{1}{2} \) particles. The basis states in the separate representation are given by:

\[
\{ \uparrow \otimes \uparrow, \uparrow \otimes \downarrow, \downarrow \otimes \uparrow, \downarrow \otimes \downarrow \} \quad (2.28)
\]

The basis states for the composite representation are

\[
\{ |1 1 \rangle, |1 0 \rangle, |1 -1 \rangle, |0 0 \rangle \} \quad (2.29)
\]

Thus the composite system will consist of a spin 1 multiplet and a spin 0 singlet. Since the basis states in both representations are eigenstates of \((\vec{S}_{\text{total}})^z\), the key lies in comparing the eigenvalues when acting on the various basis states.

Consider for example

\[
(\vec{S}_{\text{total}})^z |1 1 \rangle = \hbar |1 1 \rangle \quad (2.30)
\]

And

\[
(\vec{S}_{\text{total}})^z (\uparrow \otimes \uparrow) = (S^z \otimes \mathbb{I} + \mathbb{I} \otimes S^z)(\uparrow \otimes \uparrow) \\
= (S^z \otimes \mathbb{I})(\uparrow \otimes \uparrow) + (\mathbb{I} \otimes S^z)(\uparrow \otimes \uparrow) \\
= (S^z \uparrow)\uparrow + \uparrow \otimes (S^z \uparrow) \\
= (\hbar \frac{1}{2} \uparrow)\uparrow + \uparrow \otimes (\hbar \frac{1}{2} \uparrow) \\
= \hbar (\uparrow \otimes \uparrow) \quad (2.31)
\]

Since these basis states have the same eigenvalues, they must be equal

\[
|1 1 \rangle = \uparrow \otimes \uparrow \quad (2.32)
\]

Similarly

\[
(\vec{S}_{\text{total}})^z |1 -1 \rangle = -\hbar |1 1 \rangle, \quad (\vec{S}_{\text{total}})^z (\downarrow \otimes \downarrow) = -\hbar (\downarrow \otimes \downarrow). \quad (2.33)
\]

Hence,

\[
|1 -1 \rangle = \downarrow \otimes \downarrow \quad (2.34)
\]

But notice that

\[
(\vec{S}_{\text{total}})^z |1 0 \rangle = (\vec{S}_{\text{total}})^z |0 0 \rangle = 0 \quad (2.35)
\]

and

\[
(\vec{S}_{\text{total}})^z (\uparrow \otimes \downarrow) = (\vec{S}_{\text{total}})^z (\downarrow \otimes \uparrow) = 0 \quad (2.36)
\]

Consequently there are two states in each representation which have the same eigenvalue. In general the two from one representation will form a linear combination of the other. To find out what linear combination will be applicable we will make use of the lowering operator

\[
\vec{S}_- = S^- \otimes \mathbb{I} + \mathbb{I} \otimes S^- \quad (2.37)
\]

\(\text{6The arrow notation } \uparrow = |\frac{1}{2} \rangle \text{ and } \downarrow = |\frac{1}{2} - \frac{1}{2} \rangle \text{ is adopted for the sake of convenience.}\)
\[ S^- | s \rangle = \hbar \sqrt{s(s+1)} - m_s(m_s+1) | s \rangle (2.38) \]

Now acting with \( S_{\text{total}}^- \) on \(|1\rangle = \uparrow \otimes \uparrow\):

\[
S_{\text{total}}^- | 1 \rangle = S_{\text{total}}^- (\uparrow \otimes \uparrow) \\
= (S^- \otimes I + I \otimes S^-)(\uparrow \otimes \uparrow) \\
= (S^- \otimes I)(\uparrow \otimes \uparrow) + (I \otimes S^-)(\uparrow \otimes \uparrow) \\
= (S^- \otimes I)(\uparrow \otimes \uparrow) + \uparrow \otimes (S^- \otimes I) \\
= \hbar (\downarrow \otimes \uparrow + \uparrow \otimes \downarrow) \tag{2.39}
\]

But

\[
S_{\text{total}}^- | 1 \rangle = \hbar \sqrt{2} | 0 \rangle \tag{2.40}
\]

As a result

\[
| 0 \rangle = \frac{1}{\sqrt{2}} (\downarrow \otimes \uparrow + \uparrow \otimes \downarrow) \tag{2.41}
\]

It also follows that

\[
| 0 \rangle = \frac{1}{\sqrt{2}} (\uparrow \otimes \downarrow - \downarrow \otimes \uparrow) \tag{2.42}
\]

If one applies the raising or lowering operator to \(|00\rangle\) one should get zero, which is in fact the case.

3 The Heisenberg XXX\(_{1/2}\) Spin Chain

Now that we have established a solid understanding of the mathematical formulation of spin, we are in a position to perhaps reformulate the Heisenberg XXX\(_{1/2}\) spin chain in a more formal manner.

3.1 Definition

Consider a 1-dimensional lattice with N lattice sites and a spin \( \frac{1}{2} \) particle positioned at every lattice site which have a nearest neighbor spin-spin interaction. Each particle can have either spin up or down generating a two-dimensional local Hilbert space \( V_n \). Since we have \( N \) particles the total Hilbert space in which the states live is given by

\[
\mathcal{H} = \prod_{n=1}^{N} \otimes V_n = V \otimes \cdots \otimes V \otimes \cdots \otimes V \tag{3.1}
\]

with \( V_n = V = \mathbb{C}^2 \) for each \( n \). The elements of the two-dimensional complex vector space \( V \) can be represented by two-component vectors \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) with \( x_1, x_2 \in \mathbb{C} \).

Strictly speaking there are two possible topologies for a one dimensional chain: open or closed. Hence there are also two Heisenberg spin Hamiltonians describing the nearest neighbor interactions:

\[
H = \sum_{n=1}^{N-1} H_{n,n+1} \quad \text{(open),} \quad H = \left( \sum_{n=1}^{N-1} H_{n,n+1} \right) + H_{N,1} \quad \text{(closed)} \tag{3.2}
\]
Where $H_{i,j}$ is the two-site Hamiltonian. We shall from now on only focus on the closed chain topology which is depicted schematically in figure 1. Note that the closed chain requires the periodic boundary conditions relating the $1^{st}$ and $(N + 1)^{th}$ lattice site such that $\vec{\sigma}_{N+1} = \vec{\sigma}_1$. Keeping this in mind, the $XXX_{1/2}$ Hamiltonian is given by:

$$H = J \sum_{n=1}^{N} (\vec{S}_n \cdot \vec{S}_{n+1} - \frac{1}{4} I^{\otimes N})$$

(3.3)

Note that $J < 0$ models a ferromagnetic state while $J > 0$ models an anti-ferromagnetic state [7]. A ferromagnetic state refers to a state where all the spins are aligned parallel while an anti-ferromagnetic state refers to a state where adjacent spins are aligned anti-parallel within the domain.

The basic observables at site $n$ are given by

$$\vec{\sigma}_n = 1 \otimes \cdots \otimes I \otimes \vec{\sigma}_n \otimes \cdots \otimes I, \quad n \in \{1, 2, \ldots, N\}$$

(3.4)

with the Pauli vector and the identity matrix defined as in eq. (2.3) and eq. (2.12) respectively. These are operators which act non-trivially on the $n^{th}$ space and trivially on the rest.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{One-dimensional closed spin chain [8]}
\end{figure}

### 3.2 Symmetries

The effective size and complexity of the Hamiltonian can be reduced by looking at the symmetries of the system. The total spin vector/operator for such a system is given by

$$\vec{S}_{\text{total}} = \sum_{i=1}^{N} \vec{S}_i = \frac{1}{2} \sum_{i=1}^{N} \vec{\sigma}_i$$

(3.5)

Note that $I^{\otimes N} \equiv I \otimes \cdots \otimes I$. The constant is subtracted for convenience and only shifts the energy levels by a factor.
Similarly for \((\vec S_{\text{total}})^z\):

\[
(\vec S_{\text{total}})^z = \sum_{i=1}^{N} S_i^z = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^z
\]  

These total spin operators commute with the Hamiltonian. We now explicitly show that 
\([H, (\vec S_{\text{total}})^z] = 0\).

Consider 
\[
[H, (\vec S_{\text{total}})^z] = H(\vec S_{\text{total}})^z - (\vec S_{\text{total}})^z H
\]

\[
= (\sum_{i} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha) (\sum_{j} S_j^z) - (\sum_{j} S_j^z) (\sum_{i} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha)
\]

\[
= \sum_{i} \sum_{j} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha s_j^z - \sum_{j} \sum_{i} \sum_{\alpha} S_j^z S_i^\alpha S_{i+1}^\alpha
\]

At this point one should consider the following cases: 
(i) \(j < i\), (ii) \(j = i\), (iii) \(j = i + 1\) and (iv) \(j > i\)

(i) \(j < i\)

\[
\sum_{i} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha S_j^z = \sum_{i} \sum_{\alpha} \frac{1}{2} \times \cdots \times I \otimes \cdots \otimes I \otimes S_j^z \otimes \cdots \otimes I \otimes S_i^\alpha \otimes \cdots \otimes I
\]  

and

\[
\sum_{i} \sum_{\alpha} S_j^z S_i^\alpha S_{i+1}^\alpha = \sum_{i} \sum_{\alpha} \frac{1}{2} \times \cdots \times I \otimes \cdots \otimes I \otimes S_j^z \otimes \cdots \otimes I \otimes S_i^\alpha \otimes \cdots \otimes I
\]

Hence

\[
\sum_{i} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha S_j^z - \sum_{i} \sum_{\alpha} S_j^z S_i^\alpha S_{i+1}^\alpha = 0
\]

(ii) \(j = i\)

\[
\sum_{i} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha S_i^z = \sum_{i} \sum_{\alpha} \frac{1}{2} \times \cdots \times I \otimes S_i^\alpha \otimes S_i^z \otimes S_i^\alpha \otimes \cdots \otimes I
\]

and

\[
\sum_{i} \sum_{\alpha} S_i^z S_i^\alpha S_i^{\alpha+1} = \sum_{i} \sum_{\alpha} \frac{1}{2} \times \cdots \times I \otimes S_i^z \otimes S_i^\alpha \otimes S_i^{\alpha+1} \otimes \cdots \otimes I
\]

Hence

\[
\sum_{i} \sum_{\alpha} S_i^\alpha S_{i+1}^\alpha S_i^z - \sum_{i} \sum_{\alpha} S_i^z S_i^\alpha S_i^{\alpha+1} = \sum_{i} \sum_{\alpha} [S_i^\alpha, S_i^z] \otimes S_i^\alpha \otimes \cdots \otimes I
\]
(iii) $j = i + 1$

\[
\sum_{i}^{N} S_i^\alpha S_{i+1}^\alpha = \sum_{i}^{N} \sum_{\alpha} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^\alpha \otimes S_{i+1}^\alpha S_i^{i+1 \alpha} \otimes I \otimes \cdots \otimes I \tag{3.14}
\]

and

\[
\sum_{i}^{N} S_i^z S_{i+1}^z = \sum_{i}^{N} \sum_{\alpha} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^\alpha \otimes S_{i+1}^\alpha S_i^{i+1} \otimes I \otimes \cdots \otimes I \tag{3.15}
\]

Hence

\[
\sum_{i}^{N} S_i^\alpha S_{i+1}^\alpha S_i^z - \sum_{i}^{N} \sum_{\alpha} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^\alpha \otimes [S_i^\alpha, S_i^z] \otimes I \otimes \cdots \otimes I = 0 \tag{3.16}
\]

(iv) $j > i$

The same argument as for the case $j < i$ applies. Hence

\[
\sum_{i}^{N} \sum_{j>i}^{N} S_i^\alpha S_{j+1}^\alpha S_j^z - \sum_{i}^{N} \sum_{j>i}^{N} S_j^\alpha S_{i+1}^\alpha = 0 \tag{3.17}
\]

Now coming back to the case (ii) $i = j$. Let us write out eq. (3.13) over the values of $\alpha$.

For $x$:

\[
\sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} [S_i^x, S_i^z] \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} = -i\hbar \sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^y \otimes S_i^x \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} \tag{3.18}
\]

For $y$:

\[
\sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} [S_i^y, S_i^z] \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} = i\hbar \sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^x \otimes S_i^y \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} \tag{3.19}
\]

For $z$:

\[
\sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} [S_i^z, S_i^y] \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} = 0 \tag{3.20}
\]

Where the commutation relations of eq. (2.1) was used. In the same way writing out eq. (3.16) over the values of $\alpha$ for the case (iii) $j = i + 1$.

For $x$:

\[
\sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^x \otimes [S_i^x, S_i^z] \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} = -i\hbar \sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^y \otimes S_i^x \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} \tag{3.21}
\]

For $y$:

\[
\sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^y \otimes [S_i^y, S_i^z] \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} = i\hbar \sum_{i}^{N} \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} S_i^x \otimes S_i^y \otimes \frac{1}{i} \otimes \cdots \otimes \frac{1}{i} \tag{3.22}
\]
For $z$:

$$
\sum_{i}^{N} \mathbb{1} \otimes \cdots \otimes \mathbb{I} \otimes S_z^i \otimes [S_z^i, S_z^{i+1}] \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{1} = 0
$$

(3.23)

Adding up eq. (3.18) - (3.23) results in

$$
[H, (\vec{S}_{\text{total}})^z] = 0
$$

(3.24)

The same procedure can be used to prove that

$$
[H, (\vec{S}_{\text{total}})^x] = 0
$$

(3.25)

and

$$
[H, (\vec{S}_{\text{total}})^y] = 0
$$

(3.26)

As a result

$$
[\vec{S}_{\text{total}}, H] = 0
$$

(3.27)

The spin operators form an $su(2)$ algebra hence the spin chain also has an $su(2)$ symmetry algebra. Since we are dealing with these symmetries, the next section aims to give an introduction to Lie Groups and Lie Algebras.

4 Lie Groups and Lie Algebras

4.1 Symmetry

One of the main applications of Lie groups in modern theoretical physics is in the context of symmetries. In general, objects are symmetric if they don’t change under some transformation. Symmetries are described by some transformation of a physical system

$$
\alpha : \mathcal{M} \rightarrow \mathcal{M}
$$

(4.1)

which maps allowed configurations of the system into other allowed configurations [9].

4.2 Introductory Group Theory

Definition 1. A group is a set $G$, consisting of abstract elements, together with an associated operation $\cdot$ that satisfies the following axioms:

1. Closure under the group operation.
   
   $g_1, g_2 \in G \Rightarrow (g_1 \cdot g_2) \in G$.

2. Associativity.
   
   For any three elements $g_1, g_2, g_3 \in G \Rightarrow g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$.

3. Identity
   
   There exists an identity element $g_0$ such that $g_0 \cdot g = g \cdot g_0 = g$ for every $g \in G$.

4. Inverse
   
   For every $g \in G$ there exists an inverse element $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = g_0$. 


The group that has just been defined may be represented by the symbol \( \langle G, \cdot, g_0 \rangle \). In general there are other possible group operations on \( G \). This notation makes it explicit that the group consists of the set \( G \) with the group operation \( \cdot \) and identity element \( g_0 \). If there is no danger of confusion, the group will simply be denoted by \( G \). If the commutative law holds in a group \( G \), such a group is called an **abelian group**.

**Definition 2.** Let \( \langle G, \cdot, g_0 \rangle \) be a group. We say \( H \) is a subgroup of \( G \) if:

1. \( H \subseteq G \) and \( g_0 \in H \)
2. \( h_1, h_2 \in H \implies (h_1 \cdot h_2) \in H \) [\( H \) is closed under \( \cdot \)]
3. \( h_1 \in H \implies h_1^{-1} \in H \) [\( H \) is closed under inverses w.r.t. \( \cdot \)]

**Definition 3.** Let \( G \) and \( H \) be groups. A **homomorphism** from \( G \) to \( H \) is a map

\[
f : G \to H
\]

such that \( f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2) \) for all \( g_1, g_2 \in G \). If \( f \) is also bijective, then \( f \) is called an **isomorphism**.

4.2.1 Examples of Groups

**Example 4.2.1. The multiplicative group of real numbers**

The set of non-zero real numbers with ordinary multiplication as the group operation is one of the simplest examples of a group. By inspection one can see that the axioms (1) and (2) are satisfied. The identity element is the number 1 and each real number \( a \) has its reciprocal \( 1/a \) as its inverse (since \( a \neq 0 \)).

**Example 4.2.2. A finite matrix group**

Let the group \( G \) be the set of eight matrices with matrix multiplication as the group operation. The group elements are given by:

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
A_5 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.
\]

By explicit calculation it can be verified that axiom (1) is satisfied. Axiom (2) is automatically true for matrix multiplication. \( A_1 \) is the identity element of axiom (3) and finally axiom (4) is satisfied by:

\[
A_1^{-1} = A_1, A_2^{-1} = A_2, A_3^{-1} = A_3, A_4^{-1} = A_4, A_5^{-1} = A_6, A_6^{-1} = A_5, A_7^{-1} = A_7, A_8^{-1} = A_8
\]
Example 4.2.3. The groups $U(N)$ and $SU(N)$

$U(N)$ for $N \geq 1$ is defined as the set of all unitary matrices $u$ (i.e. $u^\dagger u = I_N$) with matrix multiplication as the group operation.

$SU(N)$ for $N \geq 2$ is defined as the subset of all unitary matrices for which $\det(u) = 1$, again with matrix multiplication as the group operation.

One can prove that these sets together with matrix multiplication do in fact form groups as follows:

Consider the set of all unitary matrices $U(N)$. Now, $(u_1u_2)^\dagger = u_2^\dagger u_1^\dagger$ and $(u_1u_2)^{-1} = u_2^{-1}u_1^{-1}$. So if $u_1$ and $u_2$ are both unitary, then $u_1u_2$ is also unitary. Hence axiom (1) is satisfied. Again axiom (2) is automatically true for matrix multiplication. Since $1_N \in U(N)$, the unit matrix is the identity element of axiom (3). Lastly if $u \in U(N)$ then $u^{-1} \in U(N)$. Hence axiom (4) is satisfied. For $SU(N)$ the same approach applies but $\det(u_1) = 1$ and $\det(u_2) = 1$. However from the properties of determinants it follows that $\det(u_1u_2) = \det(u_1)\det(u_2) = 1$ [10].

The order of a group $G$ is defined to be the number of elements in $G$ which can be finite, countably infinite or non-countably infinite. The vast majority of groups that arise in physical situations are either finite groups or “Lie groups” which are a special type of group of non-countably infinite order. The groups in examples 1 and 3 are all Lie groups. For a finite group it is useful to draw up an “operational table” with the basic format of having a row and a column for each group element. The entry in row $x$ and column $y$ is given by $x \cdot y$. For groups of infinite order the construction of such a table is impractical. Fortunately for a Lie group, the structure of the group is largely determined by a set of finite relations, namely the commutation relations between the basis elements of the corresponding real Lie algebra.

4.3 Lie Groups

Definition 4. A Lie group is a group which is also a differentiable manifold, such that the maps

$$
\mu : G \times G \rightarrow G \quad \nu : G \rightarrow G
$$

$$(g_1, g_2) \mapsto g_1 \cdot g_2 \quad g \mapsto g^{-1}
$$

are smooth. 8

Qualitatively it means that Lie groups are defined as groups whose elements $\{g_i\}$ are labeled by continuous parameters and where the multiplication law depends smoothly on the parameters.

8The subjects of topological spaces and differentiable manifolds are perhaps outside the scope of this project. This definition is given just for the sake of completeness.
4.4 Lie Algebras

A Lie algebra \( \mathcal{A} \) is a vector space over a field \( \mathbb{K} \) together with a binary operation - called a Lie bracket

\[
[\cdot, \cdot]: \mathcal{A} \times \mathcal{A} \to \mathcal{A}
\]

which has the following properties:

1. It is bilinear, \([ax + by, z] = a[x, z] + b[y, z]\)
2. It is antisymmetric, \([x, y] = -[y, x]\) (so in particular \([x, x] = 0\))
3. It satisfies the Jacobian identity, \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\)

Lie algebras are defined by commutation relations which provide the Lie bracket for all possible combinations of the elements in the algebra.

**Example 4.4.1. The \( su(2) \) algebra**

One can check explicitly that the Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

satisfy the following commutation relations:

\[
[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k \tag{4.4}
\]

With the totally antisymmetric tensor \( \epsilon^{ijk} \) defined as

\[
\epsilon^{123} = \epsilon^{231} = \epsilon^{321} = 1, \quad \epsilon^{132} = \epsilon^{213} = \epsilon^{312} = -1 \tag{4.5}
\]

The commutator of any two matrices \( A, B \) is defined as \([A, B] = AB - BA\).

Setting \( i = 1 \) and \( j = 2 \):

\[
[\sigma^1, \sigma^2] = 2i\epsilon^{12k}\sigma^k = 2i(\epsilon^{121}\sigma^1 + \epsilon^{122}\sigma^2 + \epsilon^{123}\sigma^3) = 2i\sigma^3 \tag{4.6}
\]

\[
\sigma^1 \cdot \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4.7}
\]

\[
\sigma^2 \cdot \sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{4.8}
\]

So,

\[
[\sigma^1, \sigma^2] = \sigma^1 \cdot \sigma^2 - \sigma^2 \cdot \sigma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\sigma^3 \tag{4.9}
\]

as required. Thus the relation is satisfied for \( i = 1 \) and \( j = 2 \). Similarly, it can be proven that the relation holds for \( i, j \in \{1, 2, 3\} \).
Example 4.4.2. The SU(3) algebra

One can check explicitly that the Gell-Mann matrices

\[
\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\]

satisfy the following commutation relations:

\[
[\lambda^i, \lambda^j] = 2i f^{ijk} \lambda^k \tag{4.10}
\]

With the structure constants \( f^{ijk} \) defined as

\[
f^{123} = 1, f^{458} = f^{678} = \frac{\sqrt{3}}{2},
\]

\[
f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}.
\]

Note that the indices now run from 1 to 8 and that like the \( \epsilon^{ijk} \), the \( f^{ijk} \) are antisymmetric. So all other combinations can be found by permuting indices and the structure constants for all the combinations not shown above are zero.

Setting \( i = 4 \) and \( j = 5 \):

\[
[\lambda^4, \lambda^5] = 2i f^{45k} \lambda^k
\]

\[
= 2i(f^{453} \lambda^3 + f^{458} \lambda^8)
\]

\[
= 2i\left(\frac{1}{2} \lambda^3 + \frac{\sqrt{3}}{2} \lambda^8\right)
\]

\[
= \begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix}
\]

\[
\lambda^4 \cdot \lambda^5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}
\tag{4.12}
\]

\[
\lambda^5 \cdot \lambda^4 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}
\tag{4.13}
\]
So,
\[
[\lambda^4, \lambda^5] = \lambda^4 \cdot \lambda^5 - \lambda^5 \cdot \lambda^4
= \begin{pmatrix}
i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -i
\end{pmatrix} - \begin{pmatrix}
-i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & i
\end{pmatrix} = \begin{pmatrix}
2i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2i
\end{pmatrix}
\]
(4.14)
as required. Thus the relation is satisfied for \(i = 4\) and \(j = 5\). Similarly, it can be proven that the relation holds for \(i, j \in \{1, 2, 3, 4, 5, 6, 7, 8\}\).

### 4.5 Representations

For a Lie algebra, the dimension of a matrix representation is simply the dimension of the matrices. For example the Pauli matrices \((2 \times 2)\) are a two-dimensional representation of the group SU(2). The 3D rotation matrices \((3 \times 3)\) are a three-dimensional representation of the same group. There also exists higher representations of SU(2). Note that the dimension of the group is the number of representation matrices and in the case of SU(2) it is always three. The fundamental representation is the representation which is the same as the dimension of the matrices defining the group. The Pauli matrices form the fundamental representation for the group SU(2) and the Gell-Mann matrices form the fundamental representation for SU(3). The adjoint representation of a group is the representation whose dimension (dimension of the matrices) is the same as the dimension of the group (number of matrices).

After this brief Group theory interlude we now return to the Heisenberg spin chain.

### 5 The Two-Site Problem

To get a better understanding of the \(XXX_{1/2}\) model let us for the moment examine the two-site problem explicitly.\(^9\) The basic observables are the spin operators at each site. I.e. \(\vec{\sigma}_1 \equiv \vec{\sigma} \otimes 1\) and \(\vec{\sigma}_2 \equiv 1 \otimes \vec{\sigma}\). Hence,

\[
\sigma^x_1 = \sigma^x \otimes 1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (5.1)
\]

\[
\sigma^y_1 = \sigma^y \otimes 1 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix}, \quad (5.2)
\]

\(^9\)The two-site problem is actually trivial since the distinction between the open and closed topology almost becomes redundant. Hence, in this case only, the two-site problem was evaluated as an open chain. In fact, one can verify that the open and closed topology energy spectrum only differs by a factor of 2.
\[\sigma_1^z = \sigma^z \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{5.3}\]

and

\[\sigma_2^x = I \otimes \sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.4}\]

\[\sigma_2^y = I \otimes \sigma^y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{5.5}\]

\[\sigma_2^z = I \otimes \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.6}\]

These operators act on the Hilbert space \(H = V \otimes V\) which has elements

\[x \otimes y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix}, \quad x_1, x_2, y_1, y_2 \in \mathbb{C} \tag{5.7}\]

The two-site Heisenberg spin Hamiltonian is given by

\[H_{12} = \frac{J}{4} (\vec{\sigma}_1 \cdot \vec{\sigma}_2 - I \otimes I) \]
\[= \frac{J}{4} (\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z - I \otimes I) \]
\[= \frac{J}{4} [(\sigma^x \otimes I)(I \otimes \sigma^x) + (\sigma^y \otimes I)(I \otimes \sigma^y) + (\sigma^z \otimes I)(I \otimes \sigma^z) - I \otimes I] \tag{5.8}\]
\[= \frac{J}{4} (\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z - I \otimes I) \]
\[= \frac{J}{4} (\vec{\sigma} \otimes \vec{\sigma} - I \otimes I) \]

From eq. (3.5) one can write the total spin vector as :

\[\vec{S}_{\text{total}} = \frac{1}{2} (\vec{\sigma} \otimes I + I \otimes \vec{\sigma}) \tag{5.9}\]
This implies that:

\[
\left(\vec{S}_{\text{total}}\right)^2 = \frac{1}{4} (\vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{\sigma})^2
= \frac{1}{4} [(\vec{\sigma})^2 \otimes \mathbb{I} + 2(\vec{\sigma} \otimes \vec{\sigma}) + \mathbb{I} \otimes (\vec{\sigma})^2]
= \frac{1}{4} [2(\vec{\sigma} \otimes \vec{\sigma}) + 6 \mathbb{I} \otimes \mathbb{I}]
= \frac{1}{2} \vec{\sigma} \otimes \vec{\sigma} + \frac{3}{2} \mathbb{I} \otimes \mathbb{I}
\]  
(5.10)

Rewriting the above expression somewhat leads to:

\[
\vec{\sigma} \otimes \vec{\sigma} = 2(\vec{S}_{\text{total}})^2 - 3 \mathbb{I} \otimes \mathbb{I}
\]  
(5.11)

Consequently the two-site Hamiltonian can be expressed in terms of \((\vec{S}_{\text{total}})^2\):

\[
H_{12} = \frac{J}{4} (\vec{\sigma} \otimes \vec{\sigma} - \mathbb{I} \otimes \mathbb{I}) = \frac{J}{2} [(\vec{S}_{\text{total}})^2 - 2 \mathbb{I} \otimes \mathbb{I}]
\]  
(5.12)

Let us now act with the Hamiltonian on a state \(|s m\rangle\) described in the composite representation. However, one must remember that the operator \((\vec{S}_{\text{total}})^2\) acting on a state in the composite representation is analogous to eq. (2.4). Therefore,

\[
(\vec{S}_{\text{total}})^2 |s m\rangle = s(s + 1) |s m\rangle
\]  
(5.13)

So,

\[
H_{12} |s m\rangle = \frac{J}{2} [(\vec{S}_{\text{total}})^2 - 2 \mathbb{I} \otimes \mathbb{I}] |s m\rangle
= \frac{J}{2} [s(s + 1) - 2] |s m\rangle, \quad s = 0, 1 \quad m = -s, ..., s;
\]  
(5.14)

Consequently the energy of the system is given by \(E = \frac{J}{2} [s(s + 1) - 2]\). For \(s = 1\) the energy is \(E = 0\) while for \(s = 0\) the energy is \(E = -J\).

By writing the Hamiltonian of eq. (5.8) in matrix form, the above result can be verified explicitly:

\[
H_{12} = \frac{J}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]  
(5.15)

\[
|1 1\rangle = \left|\frac{1}{2} \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} \frac{1}{2}\right\rangle = \left(\begin{array}{c} 1 \\ 0 \\
\end{array}\right) \otimes \left(\begin{array}{c} 1 \\ 0 \\
\end{array}\right) = \left(\begin{array}{c} 1 \\ 0 \\
\end{array}\right)
\]  
(5.16)

\[
|1 0\rangle = \left|\frac{1}{2} \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} -\frac{1}{2}\right\rangle + \left|\frac{1}{2} -\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2} \frac{1}{2}\right\rangle = \left(\begin{array}{c} 1 \\ 0 \\
\end{array}\right) \otimes \left(\begin{array}{c} 0 \\ 1 \\
\end{array}\right) + \left(\begin{array}{c} 0 \\ 1 \\
\end{array}\right) \otimes \left(\begin{array}{c} 1 \\ 0 \\
\end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \\
\end{array}\right)
\]  
(5.17)
\[ |1 - 1\rangle = \left( \frac{1}{2} - \frac{1}{2} \right) \otimes \left( \frac{1}{2} - \frac{1}{2} \right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \] (5.18)

\[ |0 0\rangle = \left( \frac{1}{2} \frac{1}{2} \right) \otimes \left( \frac{1}{2} \frac{1}{2} \right) \otimes \left( \frac{1}{2} \frac{1}{2} \right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \] (5.19)

One can verify by inspection that the first three states have energy \( E = 0 \) and the last state has energy \( E = -J \), which is result obtained earlier.

6 The Three-Site Problem

As a further illustration we examine the three-site problem explicitly. As with the two-site problem, the basic observables are the spin operators at each site, \( \vec{\sigma}_1 = \vec{\sigma} \otimes \mathbb{I} \otimes \mathbb{I} \), \( \vec{\sigma}_2 = \mathbb{I} \otimes \vec{\sigma} \otimes \mathbb{I} \) and \( \vec{\sigma}_3 = \mathbb{I} \otimes \mathbb{I} \otimes \vec{\sigma} \). That is,

\[ \sigma_1^x = \sigma^x \otimes \mathbb{I} \otimes \mathbb{I}, \] (6.1)

\[ \sigma_1^y = \sigma^y \otimes \mathbb{I} \otimes \mathbb{I}, \] (6.2)

\[ \sigma_1^z = \sigma^z \otimes \mathbb{I} \otimes \mathbb{I} \] (6.3)

and

\[ \sigma_2^x = \mathbb{I} \otimes \sigma^x \otimes \mathbb{I}, \] (6.4)

\[ \sigma_2^y = \mathbb{I} \otimes \sigma^y \otimes \mathbb{I}, \] (6.5)

\[ \sigma_2^z = \mathbb{I} \otimes \sigma^z \otimes \mathbb{I} \] (6.6)

and

\[ \sigma_3^x = \mathbb{I} \otimes \mathbb{I} \otimes \sigma^x, \] (6.7)

\[ \sigma_3^y = \mathbb{I} \otimes \mathbb{I} \otimes \sigma^y, \] (6.8)

\[ \sigma_3^z = \mathbb{I} \otimes \mathbb{I} \otimes \sigma^z \] (6.9)

The three-site Heisenberg Hamiltonian is given by:

\[ H = H_{12} + H_{23} + H_{31} \]

\[ = \frac{J}{4} (\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \vec{\sigma}_2 \cdot \vec{\sigma}_3 + \vec{\sigma}_3 \cdot \vec{\sigma}_1 - 3 \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}) \] (6.10)

\[ \text{10Each two-site Heisenberg Hamiltonian generalizes in the usual way for more sites.} \]
For simplicity, let us evaluate each term individually:

\[
\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z
\]
\[
= (\sigma^x \otimes I \otimes I)(I \otimes \sigma^x \otimes I) + (\sigma^y \otimes I \otimes I)(I \otimes \sigma^y \otimes I) + (\sigma^z \otimes I \otimes I)(I \otimes \sigma^z \otimes I)
\]
\[
= \sigma^x \otimes I \otimes I + \sigma^y \otimes I \otimes I + \sigma^z \otimes I \otimes I
\]
\[
= \vec{\sigma} \otimes I \otimes I
\]

\[
(6.11)
\]

\[
\vec{\sigma}_2 \cdot \vec{\sigma}_3 = \sigma_2^x \sigma_3^x + \sigma_2^y \sigma_3^y + \sigma_2^z \sigma_3^z
\]
\[
= (I \otimes \sigma^x \otimes I)(I \otimes \sigma^y \otimes I) + (I \otimes \sigma^y \otimes I)(I \otimes \sigma^z \otimes I) + (I \otimes \sigma^z \otimes I)(I \otimes \sigma^z \otimes I)
\]
\[
= I \otimes \sigma^x \otimes I + I \otimes \sigma^y \otimes I + I \otimes \sigma^z \otimes I
\]
\[
= \vec{\sigma} \otimes I \otimes I
\]

\[
(6.12)
\]

\[
\vec{\sigma}_3 \cdot \vec{\sigma}_1 = \sigma_3^x \sigma_1^x + \sigma_3^y \sigma_1^y + \sigma_3^z \sigma_1^z
\]
\[
= (I \otimes I \otimes \sigma^x)(\sigma^y \otimes I \otimes I) + (I \otimes I \otimes \sigma^y)(\sigma^z \otimes I \otimes I) + (I \otimes I \otimes \sigma^z)(\sigma^z \otimes I \otimes I)
\]
\[
= \sigma^x \otimes I \otimes I + \sigma^y \otimes I \otimes I + \sigma^z \otimes I \otimes I
\]
\[
= \vec{\sigma} \otimes I \otimes I
\]

\[
(6.13)
\]

Hence the Hamiltonian can be written as:

\[
H = \frac{J}{4} (\vec{\sigma} \otimes \vec{\sigma} \otimes I + I \otimes \vec{\sigma} \otimes I + \vec{\sigma} \otimes I \otimes I - 3I \otimes I \otimes I)
\]

\[
(6.14)
\]

The total spin vector for this system is given by:

\[
\vec{S}_{\text{total}} = \frac{1}{2} (\vec{\sigma} \otimes I \otimes I + I \otimes \vec{\sigma} \otimes I + I \otimes I \otimes \vec{\sigma})
\]

\[
(6.15)
\]

This implies that:

\[
(\vec{S}_{\text{total}})^2 = \frac{1}{4} (\vec{\sigma} \otimes I \otimes I + I \otimes \vec{\sigma} \otimes I + \vec{\sigma} \otimes I \otimes I)^2
\]
\[
= \frac{1}{4} [(\vec{\sigma})^2 \otimes I \otimes I + I \otimes (\vec{\sigma})^2 \otimes I \otimes I + \vec{\sigma} \otimes (\vec{\sigma})^2 \otimes I \otimes I + 2(\vec{\sigma} \otimes \vec{\sigma} \otimes I) + 2(\vec{\sigma} \otimes I \otimes \vec{\sigma}) + 2(I \otimes \vec{\sigma} \otimes \vec{\sigma})]
\]
\[
= \frac{1}{4} [2(\vec{\sigma} \otimes \vec{\sigma} \otimes I) + 2(\vec{\sigma} \otimes I \otimes \vec{\sigma}) + 2(I \otimes \vec{\sigma} \otimes \vec{\sigma}) + 9I \otimes I \otimes I]
\]

\[
(6.16)
\]

Rewriting the above expression leads to:

\[
(\vec{\sigma} \otimes \vec{\sigma} \otimes I) + (\vec{\sigma} \otimes I \otimes \vec{\sigma}) + (I \otimes \vec{\sigma} \otimes \vec{\sigma}) = 2(\vec{S}_{\text{total}})^2 - \frac{9}{2}(I \otimes I \otimes I)
\]

\[
(6.17)
\]

Substituting eq. (6.17) into the Hamiltonian of eq. (6.14):

\[
H = \frac{J}{4} [2(\vec{S}_{\text{total}})^2 - \frac{15}{2}(I \otimes I \otimes I)]
\]

\[
(6.18)
\]
Acting with the Hamiltonian on a state $|s \, m_s\rangle$ described in the composite representation and taking eq. (5.13) into consideration

$$H |s \, m_s\rangle = \frac{J}{4} [2(S_{\text{total}})^2 - \frac{15}{2} (I \otimes I)] |s \, m_s\rangle$$

$$= \frac{J}{4} [2s(s + 1) - \frac{15}{2}] |s \, m_s\rangle, \quad s = \frac{3}{2}, \frac{1}{2}, m_s = -s...s;$$

Equation (6.19)

As a result, the energy of the system is given by $E = \frac{J}{4} [2s(s + 1) - \frac{15}{2}]$. If $s = \frac{3}{2}$ then $E = 0$ and if $s = \frac{1}{2}$ then $E = -\frac{3}{2} J$.

It is clear that it becomes increasingly difficult to explicitly solve for the energy spectrum as $N$ increases. The problem thus lies in diagonalizing the Hamiltonian for larger values of $N$. Computational methods are not very practical since $H$ is a $2^N \times 2^N$ matrix. An elegant alternative approach is the algebraic Bethe Ansatz which is discussed in the next section.

7 Algebraic Bethe Ansatz

The objective of this section is to show that the Heisenberg Hamiltonian can be expressed in terms of an $R$ matrix which it is a solution of the Yang-Baxter equation - which leads to the integrability of the model. The most convenient approach will be to work backwards and construct the so called transfer matrix - a one parameter commutative family of operators acting on the full space of states of the Heisenberg spin chain. Lastly we then show that the Heisenberg Hamiltonian is among these operators.

7.1 The Lax operator

The basic tool of the algebraic Bethe Ansatz approach is the so-called Lax operator $L$. Consider once again a chain with $N$ sites and a corresponding Hilbert space

$$\mathcal{H} = \bigotimes_{n=1}^{N} V_n$$

(7.1)

with $V_n = \mathbb{C}^2$ as in eq. (3.1). These spaces we call the quantum spaces. To these spaces we add an additional auxiliary space $V_a$. In our case $V_a = \mathbb{C}^2$. The definition of the Lax operator involves the local quantum space $V_n$ and the auxiliary space $V_a$ such that

$$L : V_n \otimes V_a \rightarrow V_n \otimes V_a$$

(7.2)

and has the form

$$L_{n,a}(u) = u(I_n \otimes I_a) + i \sum_\alpha S_n^\alpha \otimes \sigma_a^\alpha, \quad \alpha = x, y, z$$

(7.3)

where $I_n, S_n^\alpha$ are operators in $V_n$ and $I_a, \sigma_a^\alpha$ are operators in $V_a$ and lastly $\sigma_a^\alpha$ are the Pauli matrices as defined in eq. (2.3). The parameter $u$ is a complex number and is called the
spectral parameter. Alternatively $L_{n,a}(u)$ can be written as a $2 \times 2$ matrix.

\[
L_{n,a}(u) = u(I_n \otimes I_a) + iS_n^x \otimes \sigma_a^x + iS_n^y \otimes \sigma_a^y + iS_n^z \otimes \sigma_a^z
\]

\[
= \begin{pmatrix} uI_n & 0 \\ 0 & uI_n \end{pmatrix} + iS_n^x \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + iS_n^y \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + iS_n^z \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}
\]

\[
= \begin{pmatrix} uI_n + iS_n^x & \sigma_a^x \\ iS_n^x - S_n^y & uI_n - iS_n^z \end{pmatrix}
\]

\[(7.4)\]

where the relation

\[
iS_n^\pm = iS_n^x \mp S_n^y
\]

\[(7.5)\]

was used to pass from line 3 to line 4. Note that this is shorthand notation since a $2 \times 2$ matrix should actually have a basis of two dimensional vectors. So this is a $2 \times 2$ matrix in the auxiliary space $V_a$ and the matrix entries are operators acting on the physical Hilbert space $V_n$. The interpretation of the auxiliary space is simply the space from which the matrix structure of $L$ originates [2].

Let us introduce the permutation operator :

\[
P_{n,a} = \frac{1}{2}(I_n \otimes I_a + \sum_\alpha \sigma_n^\alpha \otimes \sigma_a^\alpha)
\]

\[(7.6)\]

As with the Lax operator, the permutation operator can be written as as a $2 \times 2$ matrix in the auxiliary space $V_a$

\[
P_{n,a} = \frac{1}{2}(I_n \otimes I_a + \sum_\alpha \sigma_n^\alpha \otimes \sigma_a^\alpha)
\]

\[
= \frac{1}{2} \left[ \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} + \sigma_n^x \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sigma_n^y \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \sigma_n^z \otimes \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right]
\]

\[(7.7)\]
Using the permutation operator and the fact that $S_n^\alpha = \frac{1}{2} \sigma_n^\alpha$ one can rewrite the Lax operator as:

$$L_{n,a}(u) = u(I_n \otimes I_a) + i \sum_\alpha S_n^\alpha \otimes \sigma_a^\alpha$$

$$= u(I_n \otimes I_a) + i \frac{1}{2} \sum_\alpha \sigma_n^\alpha \otimes \sigma_a^\alpha$$

$$= u(I_n \otimes I_a) - \frac{i}{2} (I_n \otimes I_a) + \frac{i}{2} (I_n \otimes I_a) + i \frac{1}{2} \sum_\alpha \sigma_n^\alpha \otimes \sigma_a^\alpha$$

$$= (u - \frac{i}{2}) (I_n \otimes I_a) + \frac{i}{2} (I_n \otimes I_a + \sum_\alpha \sigma_n^\alpha \otimes \sigma_a^\alpha)$$

$$= (u - \frac{i}{2}) I_{n,a} + iP_{n,a}$$

(7.8)

where $I_{n,a} = I_n \otimes I_a$. One can easily check that the permutation operator indeed permutes the factors in the tensor product by calculating $P$ - by calculating the sum of tensor products of the Pauli matrices with respect to the standard base, i.e.

$$P = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix} + \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(7.9)

So if $x = \begin{pmatrix} a \\ b \end{pmatrix}$ and $y = \begin{pmatrix} c \\ d \end{pmatrix}$ then $x \otimes y = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$ and $y \otimes x = \begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix}$. Therefore

$$P(x \otimes y) = P \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = \begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix} = y \otimes x, \quad P(y \otimes x) = P \begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = x \otimes y$$

(7.10)

Hence $P$ only moves the off-diagonal components and is therefore a permutation.

### 7.2 Fundamental Commutation Relations and Monodromy Matrix

Now, say one assigns two Lax operators $L_{n,a_1}(u_1)$ and $L_{n,a_2}(u_2)$ with two different auxiliary spaces $V_{a_1}$ and $V_{a_2}$ to the same local quantum state $V_n$. The product of these two operators is then a triple tensor product in $V_n \otimes V_{a_1} \otimes V_{a_2}$. The relation between these two Lax operators can be written in the following compact form which is called the fundamental commutation relation.

$$R_{a_1,a_2}(u_1 - u_2)L_{n,a_1}(u_1)L_{n,a_2}(u_2) = L_{n,a_2}(u_2)L_{n,a_1}(u_1)R_{a_1,a_2}(u_1 - u_2)$$

(7.11)

where the operator $R$ - called the quantum R-matrix - relates the permutations and spectral parameters of the two auxiliary spaces and is given by

$$R_{a_1,a_2}(u_1 - u_2) = (u_1 - u_2)I_{a_1,a_2} + iP_{a_1,a_2}$$

(7.12)
where $R$ acts in $V_{a_1} \otimes V_{a_2}$. In view of eq. (7.7) one can also write $R$ as a $2 \times 2$ matrix in either auxiliary space - i.e.

$$R_{a_1,a_2} = \begin{pmatrix} (u + \frac{i}{2})I_{a_1} + iS_{a_1}^- & iS_{a_2}^- \\ iS_{a_1}^- & (u + \frac{i}{2})I_{a_2} - iS_{a_2}^+ \end{pmatrix}_{a_1} = \begin{pmatrix} (u + \frac{i}{2})I_{a_1} + iS_{a_1}^- & iS_{a_2}^- \\ iS_{a_1}^- & (u + \frac{i}{2})I_{a_2} - iS_{a_2}^+ \end{pmatrix}_{a_2}$$

(7.13)

The fundamental commutation relation of eq. (7.11) is most easily checked by using the expression for the Lax operators in terms of the permutation operators.

$$LHS = \left( (u_1 - u_2)I_{a_1,a_2} + iP_{a_1,a_2} \right)( (u - \frac{i}{2})I_{a_1} + iP_{a_1,a_1}) ( (u - \frac{i}{2})I_{a_2} + iP_{a_1,a_2} )$$

$$= \left( (u_2 - \frac{i}{2})I_{a_2,a_2} + iP_{a_2,a_2} \right)( (u_1 - u_2)I_{a_1,a_2} + iP_{a_1,a_2}) ( (u_1 - u_2)I_{a_1,a_2} + iP_{a_1,a_2} )$$

(7.14)

$$= RHS$$

Where the following properties of permutations were used

$$P_{n,a_1}P_{n,a_2} = P_{a_1,a_2}P_{n,a_1} = P_{n,a_2}P_{a_2,a_1}, \quad P_{a,b} = P_{b,a}$$

(7.15)

From the fundamental commutation relation it can be shown that the $R$ matrix needs to satisfy the quantum Yang-Baxter equation. Consider the product of the following $L$ operators

$$L_{n,1}L_{n,2}L_{n,3} = R_{12}^{-1}L_{n,2}L_{n,1}L_{n,3}R_{12}$$

$$= R_{12}^{-1}R_{13}^{-1}L_{n,2}L_{n,1}R_{13}R_{12}$$

$$= R_{12}^{-1}R_{13}^{-1}R_{23}^{-1}L_{n,2}L_{n,1}R_{23}R_{13}R_{12}$$

$$= (R_{23}R_{13}R_{12})^{-1}L_{n,2}L_{n,1}(R_{23}R_{13}R_{12})$$

(7.16)

and

$$L_{n,1}L_{n,2}L_{n,3} = R_{23}^{-1}L_{n,1}L_{n,3}L_{n,2}R_{23}$$

$$= R_{23}^{-1}R_{13}^{-1}L_{n,3}L_{n,1}L_{n,2}R_{13}R_{23}$$

$$= (R_{12}R_{13}R_{23})^{-1}L_{n,3}L_{n,1}(R_{12}R_{13}R_{23})$$

(7.17)

For both these relations to hold we must impose that the $R$ matrix must satisfy:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

(7.18)

which is the quantum Yang-Baxter equation.$^{11}$

The **monodromy matrix** $T_{N,a}(u)$ is defined as the product of the subsequent $L$ operators

$$T_{N,a}(u) = L_{N,a}(u) \cdots L_{1,a}(u)$$

(7.19)

$^{11}$Refer to appendix B.
The monodromy matrix can be seen as a $2 \times 2$ matrix acting on the auxiliary space $V_a$ whose entries are operators in the full physical/quantum Hilbert space. Accordingly

$$T_{N,a}(u) = \begin{pmatrix} u \|_N + iS_\|_N^+ & iS_\|_N^- \\ iS_N^+ & u \|_N - iS_N^- \end{pmatrix}_a \cdots \begin{pmatrix} u \|_1 + iS_1^+ & iS_1^- \\ iS_1^+ & u \|_1 - iS_1^- \end{pmatrix}_a = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \quad (7.20)$$

The monodromy matrix obeys the fundamental relation

$$R_{a_1,a_2}(u_1 - u_2) T_{N,a_1}(u_1) T_{N,a_2}(u_2) = T_{N,a_2}(u_2) T_{N,a_1}(u_1) R_{a_1,a_2}(u_1 - u_2) \quad (7.21)$$

We will prove that eq. (7.21) holds for $N = 2$ :

$$LHS = R_{a_1,a_2}(u_1 - u_2) T_{2,a_1}(u_1) T_{2,a_2}(u_2)$$
$$= R_{a_1,a_2}(u_1 - u_2) L_{2,a_1}(u_1)L_{1,a_1}(u_1)L_{2,a_2}(u_2)L_{1,a_2}(u_2)$$
$$= [R_{a_1,a_2}(u_1 - u_2)L_{2,a_1}(u_1)L_{2,a_2}(u_2)] L_{1,a_1}(u_1)L_{1,a_2}(u_2)$$
$$= L_{2,a_2}(u_2) L_{2,a_1}(u_1) [R_{a_1,a_2}(u_1 - u_2) L_{1,a_1}(u_1)L_{1,a_2}(u_2)] \quad (7.22)$$
$$= L_{2,a_2}(u_2) L_{2,a_1}(u_1) L_{1,a_2}(u_2) L_{1,a_1}(u_1) R_{a_1,a_2}(u_1 - u_2)$$
$$= R_{a_1,a_2}(u_1 - u_2) L_{2,a_1}(u_1)L_{2,a_2}(u_2)L_{1,a_1}(u_1)L_{1,a_2}(u_1)\quad (7.23)$$

$$RHS$$

where the fundamental commutation relation of eq. (7.11) and the fact that $L_{n_1,a_1}$ commutes with $L_{n_2,a_2}$ for $n_1 \neq n_2$ was used throughout. The procedure used in the above proof can be extended to the case of an arbitrary $N$ via mathematical induction.

The transfer matrix is defined by taking the trace over the auxiliary space $V_a$

$$t(u) = tr_a T_{N,a}(u) = tr_a \left( \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \right) = A(u) + D(u) \quad (7.24)$$

Now by taking the doubletrace $tr_{a_1} tr_{a_2}$ over the two auxiliary spaces $V_{a_1}$, $V_{a_2}$ of eq. (7.21) we derive

$$t(u_1) t(u_2) = t(u_2) t(u_1), \quad \text{I.e.} \quad [t(u_1), t(u_2)] = 0 \quad (7.25)$$

The result obtained is that the transfer matrix commutes with itself for different values of the spectral parameter $u$. One can now expand the monodromy matrix in a power series around any point $u_0$ of the complex plane. In doing so one generates an infinite set of linearly independent commuting operators acting on the full quantum space. This formally proves the integrability of the model.

**7.3 The Monodromy and the Hamiltonian**

It remains to be shown that the Hamiltonian belongs to the family of the transfer matrices. Let us now expand the monodromy matrix around the point $u = \frac{i}{2}$ to first order :

$$T_{N,a}(\frac{i}{2}) = i^N P_{N,a} P_{N-1,a} \cdots P_{1,a} \quad (7.26)$$

$$= i^N P_{1,2} P_{2,3} \cdots P_{N-1,N} P_{N,a}$$
where the properties of permutations were used to pass to the second line. Taking the trace over the auxiliary space becomes simple since it only appears in the last term and

\[ tr_a P_{N,a} = tr_a \left( \frac{1}{2} \left( \mathbb{1}_n + \sigma^x_n \frac{\sigma^y_n - \sigma^z_n}{\sigma^x_n + i\sigma^y_n} \right) \right) = \mathbb{1}_N \]  

(7.26)

So,

\[ t\left( \frac{i}{2} \right) = tr_a T_{N,a}\left( \frac{i}{2} \right) = \mathbb{1}_N \]

(7.27)

So,

\[ \mathcal{U} = i^{-N} t\left( \frac{i}{2} \right) = P_{1,2} P_{2,3} \cdots P_{N-1,N} \]

(7.28)

Now to see that \( \mathcal{U} \) is a shift operator in the full quantum Hilbert space \( \mathcal{H} \) we use the following property of the permutation operator

\[ P_{n,1,n+1} X_n = X_{n+1} \]

(7.29)

So this permutation moves \( X \) one step back. Furthermore it can be shown that \( \mathcal{U} \) is unitary. From these facts it can be concluded that \( \mathcal{U} \) is the shift operator which is related to the momentum operator as

\[ \mathcal{U} = e^{i P} \]

(7.30)

For the next order in the expansion of the transfer matrix we take the derivative of the monodromy matrix at the point \( u = \frac{i}{2} \):

\[ \frac{d T_{N,a}(u)}{du} \bigg|_{u=\frac{i}{2}} = i^{N-1} \sum_n P_{N,a} \cdots P_{n-1,a} P_{n+1,a} \cdots P_{1,a} \]

(7.31)

So,

\[ \frac{dt(u)}{du} \bigg|_{u=\frac{i}{2}} = \frac{d(tr_a T_{N,a}(u))}{du} \bigg|_{u=\frac{i}{2}} \]

(7.32)

\[ = i^{N-1} \sum_n P_{1,2} P_{2,3} \cdots P_{n-1,n+1} \cdots P_{N-1,N} P_{N,a} \]

where eq. (7.26) was used again. But,

\[ \frac{d}{du} t(u) t(u)^{-1} \bigg|_{u=\frac{i}{2}} = \frac{d}{du} \ln(t(u)) \bigg|_{u=\frac{i}{2}} \]

(7.33)

\[ = \frac{1}{i} \sum_n P_{n,n+1} \]
Using eq. (3.3) and eq. (7.6) one can write the Hamiltonian as

\[ H = \frac{J}{2} \left( \sum_n P_{n,n+1} - N \right) \]

(7.34)

And

\[ H = \frac{J}{2} \left[ i \frac{d}{du} \ln(t(u)) \right]_{u=\frac{i}{2}} - N \]

(7.35)

Thus we have shown that the Hamiltonian belongs to the family of \( N - 1 \) commuting operators generated by the trace of the monodromy \( T_{N,a} \). I.e the Hamiltonian is part of the trace of the monodromy, as was required. As a result, we see that the Hamiltonian commutes with the transfer matrix.

\[ [H, t(u)] = 0 \]

(7.36)

### 7.4 Diagonalizing the Hamiltonian

The only task that is left is to diagonalize the Hamiltonian. We shall in fact diagonalize the transfer matrix. In this section, only a summary of the diagonalization procedure is given since it is already described comprehensively in numerous papers \[11\],[12],[13]\.

Recall that the monodromy matrix \( T_{N,a} \) is a \( 2 \times 2 \) matrix in the auxiliary space \( V_a \) and whose entries are operators in the full quantum space.

\[ T_{N,a}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \]

(7.37)

A set of algebraic relations among these four operators can be found from the fundamental commutation relation (7.11). The relations include

\[ [B(u_1), B(u_2)] = 0 \]

(7.38)

\[ A(u_1)B(u_2) = f(u_1 - u_2)B(u_2)A(u_1) + g(u_1 - u_2)B(u_1)A(u_2) \]

(7.39)

\[ D(u_1)B(u_2) = h(u_1 - u_2)B(u_2)D(u_1) + k(u_1 - u_2)B(u_1)D(u_2) \]

(7.40)

where \( f, g, h, k \) are given by

\[ f(u) = u - \frac{i}{u}, \quad g(u) = \frac{i}{u}, \quad h(u) = \frac{u + i}{u}, \quad k(u) = -\frac{i}{u} \]

(7.41)

Now, it can be shown that the ferromagnetic vacuum state \( \omega_+ \) with all spins up

\[ \omega_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

(7.42)

is an eigenstate of \( A(u) \) and \( D(u) \) and is annihilated by \( C(u) \). I.e

\[ A(u)\omega_+ = (u + \frac{i}{2})^N \omega_+, \quad D(u)\omega_+ = (u - \frac{i}{2})^N \omega_+, \quad C(u)\omega_+ = 0 \]

(7.43)
One can use the $B(u)$ operators to construct the so-called Bethe state

$$|u_1, u_2, ..., u_M\rangle = B(u_1) \cdots B(u_M) \omega_+ \quad (7.44)$$

Now, using the algebraic relations of eq. (7.38), eq. (7.39), eq. (7.40) and the properties of the ferromagnetic vacuum state eq. (7.43) it can be shown that the Bethe state is an eigenstate of the transfer matrix $t(u)$

$$t(u) |u_1, u_2, ..., u_M\rangle = \Lambda(u; u_1, u_2, ..., u_M) |u_1, u_2, ..., u_M\rangle \quad (7.45)$$

with eigenvalue

$$\Lambda(u; u_1, u_2, ..., u_M) = (u + \frac{i}{2})^N \prod_{\beta=1}^{M} \frac{u - u_\beta + i}{u - u_\beta} + (u - \frac{i}{2})^N \prod_{\beta=1}^{M} \frac{u - u_\beta - i}{u - u_\beta} \quad (7.46)$$

if $\{u_1, ..., u_M\}$ are distinct and satisfy the so-called Bethe Ansatz equations:

$$\left(\frac{u_\beta + \frac{i}{2}}{u_\beta - \frac{i}{2}}\right)^N = \prod_{\beta=1}^{M} \frac{u_\beta - u_\gamma + i}{u_\beta - u_\gamma - i}, \quad \beta = 1, ..., M \quad (7.47)$$

It follows from eq. (7.46) and eq. (7.35) that the energy eigenvalues for the system are

$$E = -\frac{J}{2} \sum_{\beta=1}^{M} \frac{1}{u_\beta^2 + \frac{1}{4}} \quad (7.48)$$

8 \text{ su}(2) Symmetry of the $XXX_{1/2}$ Model

To describe the Heisenberg spin chain in terms of $\text{su}(2)$ symmetry, one should show that the transfer matrix commutes with the total spin operators. A relation that was also used in section 7 (although it wasn’t formally defined) is that the monodromy can be expanded to a polynomial in $u$ of order $N$.

$$T_{N,a_1}(u_1) = (u_1)^N + i(u_1)^{N-1} \sum_\alpha (S^\alpha \otimes \sigma^\alpha) + ... \quad (8.1)$$

The expansion of the monodromy matrix in section 7 was done around the point $u = \frac{i}{2}$, since at this point the Lax operator becomes rather simple. However there is a second case where this happens, namely at $u \to \infty$. By letting $u_2$ tend to infinity in the fundamental commutation relation of the monodromy of eq. (7.21) and using eq. (8.1) we get

$$\text{LHS} = ((u_1 - u_2)I_{a_1, a_2} + iP_{a_1, a_2}) T_{N,a_1}(u_1) [(u_2)^N + i(u_2)^{N-1} \sum_\alpha (S^\alpha \otimes \sigma^\alpha)] \quad (8.2)$$

$$\text{RHS} = [(u_2)^N + i(u_2)^{N-1} \sum_\alpha (S^\alpha \otimes \sigma^\alpha)] T_{N,a_1}(u_1)((u_1 - u_2)I_{a_1, a_2} + iP_{a_1, a_2}) \quad (8.3)$$
Equating both sides results in the cancellation of the leading order terms and the following relation is obtained

\[ [T_{N,a_1}(u_1), \frac{1}{2} \sigma^\alpha + S^\alpha] = 0 \] (8.4)

From the properties of the commutator it follows that

\[ [S^\alpha, T_{N,a_1}(u_1)] = \frac{1}{2} [T_{N,a_1}(u_1), \sigma^\alpha] \] (8.5)

One can for example find that

\[ [S^z, B(u_1)] = -B(u_1), \quad [S^+, B(u_1)] = A(u_1) - D(u_1) \] (8.7)

and

\[ [S^z, A(u_1)] = [S^z, D(u_1)] = 0 \] (8.8)

\[ [S^+, A(u_1)] = [-S^+, D(u_1)] = -C(u_1) \] (8.9)

\[ [S^-, A(u_1)] = [-S^-, D(u_1)] = B(u_1) \] (8.10)

So \(^{12}\)

\[ [S^\alpha, t(u_1)] = [S^\alpha, A(u_1) + D(u_1)] \]
\[ = [S^z, A(u_1) + D(u_1)] + [S^+, A(u_1) + D(u_1)] + [S^-, A(u_1) + D(u_1)] \]
\[ = (C(u_1) - C(u_1)) + (B(u_1) - B(u_1)) \]
\[ = 0 \] (8.11)

as was required. So in summarizing, since the Heisenberg spin chain has \( su(2) \) symmetry, the eigenstates of the Hamiltonian can be split into irreducible representations of this algebra [2].

9 Conclusion

What remains is to perhaps generalize the XXX model to a spin chain of arbitrary spin \( s \in \{1, 1/2, 3/2, \ldots\} \). However, one has to be very careful in writing down the XXX Hamiltonian for a general spin chain. For example in obtaining the XXX Hamiltonian,

\[ H = \sum_{n=1}^{N} [\vec{S}_n \cdot \vec{S}_{n+1} - (\vec{S}_n \cdot \vec{S}_{n+1})^2] \] (9.1)

the naive substitution of spin 1 operators by those of spin 1/2 operators

\[ S^x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rightarrow \quad S^x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ S^y = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad S^y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ S^z = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rightarrow \quad S^z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[^{12}\text{Note that the indices that } \alpha \text{ run over was changed from } x, y, z \text{ to } +, -, z.\]
into eq. (1.1) will not result in an integrable Hamiltonian [12]. An understanding between the connection of integrable Hamiltonians and Lax operators is therefore needed. To maintain integrability one should find it as a member of the commuting family of operators, generating function for which will be trace of an appropriate monodromy of the family of local Lax operators, satisfying the fundamental commutation relations - via the algebraic Bethe Ansatz. It is therefore impossible to try and give a general Hamiltonian without considering the methods used in the $XXX_{1/2}$ model. Generally, what one can say is that the physical Hilbert space of the $XXX$ model is given by

$$\mathcal{H} = \prod_{n=1}^{N} \mathbb{C}^{2^{r+1}}.$$  \hspace{1cm} (9.5)

Lastly, one of the important applications of Heisenberg spin chains in quantum field theory that can be mentioned refers to where some operators in quantum field theory can be mapped to states in a specific spin chain model and the dimension of the operators can be mapped to the energy in the spin chain. This is a relatively new field of study and is known as AdS/CFT Integrability. Many recent advances have been made as is described in [14].

A Multiparticle systems and Tensor products

The general idea of this section is to get familiar with the tools needed to describe a system that contains more than one particle. In particular these tools will be needed in order to understand the addition of angular momenta [15]. Consider two particles. A description of the quantum mechanics and set of operators associated with each particle is given below :

1. Particle 1:
   - It’s quantum mechanics is described by a complex vector space $V$ and has associated operators $T_1, T_2, T_3, ...$

2. Particle 2:
   - It’s quantum mechanics is described by a complex vector space $W$ and has associated operators $S_1, S_2, S_3, ...$

Now that the two particles individually are identified, a description of the quantum states of this two-particle system is needed. It is thus necessary to introduce a new operation called the tensor product, denoted as $\otimes$, such that the element $v \otimes w$ with $v \in V$ and $w \in W$
can be viewed as a vector in a new vector space $V \otimes W$ which carries the description of the quantum states of the composite two-particle system. Below is given some of the basic properties of the tensor product.

1. $(av) \otimes w = v \otimes (aw) = a(v \otimes w), a \in \mathbb{C}$
2. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
3. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

Next it is required to understand how to construct operators that act in the vector space $V \otimes W$. Let $T$ be a operator in $V$ and $S$ be an operator in $W$. One can construct an operator $T \otimes S$ which is defined to act as follows :

$$(T \otimes S)(v \otimes w) \equiv Tv \otimes Sw.$$  \hfill (A.1)

Suppose now that we have an operator $T$ in $V$ that acts on the first particle in the tensor product space $V \otimes W$ even though we do not want to act on the $W$ part. The operator $T$ that acts on a single vector space needs to be upgraded into one that acts on the tensor product vector space. Hence

$$T \rightarrow T \otimes I, \quad T \otimes I \equiv Tv \otimes w.$$  \hfill (A.2)

where $I$ is called the identity operator(matrix). The Tensor product can also be defined using matrices.

**Definition :** The tensor product for the $2 \times 2$ matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ is defined as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$  \hfill (A.3)

This definition generalizes to matrices of different sizes in the usual way.

**B  The Yang - Baxter Equation**

The Yang-Baxter equation has been studied for over two decades as the master equation in integrable models in statistical models and quantum field theory.

Consider the R matrix defined as :

$$R(\lambda) = \lambda I \otimes I + i\mathcal{P} = \begin{pmatrix} \lambda + i & 0 & 0 & 0 \\ 0 & \lambda & i & 0 \\ 0 & i & \lambda & 0 \\ 0 & 0 & 0 & \lambda + i \end{pmatrix} = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c & 0 \\ 0 & c & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}.$$  \hfill (B.1)
Where the following functions are defined,

\[ a(\lambda) = \lambda + i, \quad b(\lambda) = \lambda, \quad c = i \]  \hspace{1cm} (B.2)

And the permutation matrix \( P \) is defined as

\[
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\] \hspace{1cm} (B.3)

In one dimensional quantum systems, \( R \) is called the scattering matrix and if it satisfies the Yang-Baxter equation then the system is said to be integrable. One should regard \( R(\lambda) \) as an operator which acts on \( V \otimes V \). The variable \( \lambda \) is called the spectral parameter.

Now define \( R_{12}(\lambda) \) as :

\[
R_{12}(\lambda) = R(\lambda) \otimes I = \begin{pmatrix}
a(\lambda) & 0 & 0 & 0 \\
0 & b(\lambda) & c & 0 \\
0 & c & b(\lambda) & 0 \\
0 & 0 & 0 & a(\lambda) \\
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b(\lambda) & 0 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & b(\lambda) & 0 & c & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & b(\lambda) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 \\
\end{pmatrix}
\] \hspace{1cm} (B.4)

This is an operator on \( V \otimes V \otimes V \) and acts non-trivially on the first and second spaces and acts trivially on the third.
Next define $R_{23}(\lambda)$ as:

$$R_{23}(\lambda) = \mathbb{I} \otimes R(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c & 0 \\ 0 & c & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & c & 0 & 0 & 0 & 0 & 0 \\ 0 & c & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\lambda) & c & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda) \end{pmatrix}$$

(B.5)

This is an operator on $V \otimes V \otimes V$ and acts non-trivially on the second and third spaces and acts trivially on the first.

Lastly it is required to define an operator $R_{13}(\lambda)$ on $V \otimes V \otimes V$ which acts non-trivially on the first and third spaces and acts trivially on the second. This can be accomplished as follows:

Define

$$R_{13}(\lambda) = \mathcal{P}_{23} R_{12}(\lambda) \mathcal{P}_{23}$$

(B.6)

where $R_{12}(\lambda)$ is given by eq (B.4) and $\mathcal{P}_{23}$ is given by:

$$\mathcal{P}_{23} = \mathbb{I} \otimes \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(B.7)

Hence it follows that,

$$R_{13}(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & a(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b(\lambda) & 0 & 0 & c & 0 \\ 0 & c & 0 & 0 & b(\lambda) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & 0 \end{pmatrix}$$

(B.8)
The Yang-Baxter equation is given by:

\[ R_{12}(\lambda - \hat{\lambda})R_{13}(\lambda)R_{23}(\hat{\lambda}) = R_{23}(\hat{\lambda})R_{13}(\lambda)R_{12}(\lambda - \hat{\lambda}) \]  

(B.9)

Now, let us show that this relation is satisfied by the R matrix. Consider,

\[
R_{12}(\lambda - \hat{\lambda}) = \begin{pmatrix}
    a(\lambda - \lambda') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & a(\lambda - \lambda') & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & b(\lambda - \lambda') & 0 & c & 0 & 0 & 0 \\
    0 & 0 & 0 & b(\lambda - \lambda') & 0 & c & 0 & 0 \\
    0 & 0 & c & 0 & b(\lambda - \lambda') & 0 & 0 & 0 \\
    0 & 0 & 0 & c & 0 & b(\lambda - \lambda') & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & a(\lambda - \lambda') & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda - \lambda')
\end{pmatrix},
\]

(B.10)

\[
R_{23}(\lambda') = \begin{pmatrix}
    a(\lambda') & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & b(\lambda') & c & 0 & 0 & 0 & 0 & 0 \\
    0 & c & b(\lambda') & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & a(\lambda') & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & b(\lambda') & c & 0 & 0 \\
    0 & 0 & 0 & 0 & c & b(\lambda') & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & a(\lambda') & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & a(\lambda')
\end{pmatrix},
\]

(B.11)

Substituting the above expressions in the LHS and RHS of eq. (B.9) and comparing:

**LHS** is given by

\[
\begin{pmatrix}
    (\lambda - \lambda' + i)(\lambda + i)(\lambda' + i) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda
\end{pmatrix}
\]

(B.12)

**RHS** is given by

\[
\begin{pmatrix}
    (\lambda - \lambda' + i)(\lambda + i)(\lambda' + i) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & (\lambda - \lambda' + i)\lambda
\end{pmatrix}
\]

(B.13)

\(^{13}\)Numerous other solutions to the Yang-Baxter equation are known, however this R matrix is probably the simplest solution.
So,

\[
LHS - RHS = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]  \tag{B.14}

Therefore \( LHS = RHS \) which shows that the Yang-Baxter equation is in fact satisfied by this \( R \) matrix.

References


http://hitoshi.berkeley.edu/221a/tensorproduct.pdf.


